EXERCISE – IV   HINTS & SOLUTIONS

Sol.1 $Rr(\sin A + \sin B + \sin C) = \Delta$

$$\text{L.H.S.} = r\left(\frac{a}{2}+\frac{b}{2}+\frac{c}{2}\right) = rS = \Delta$$

Sol.2 $2R\cos A = 2R + r - r_1$

$$\text{R.H.S.} = 2R + 4R\sin\frac{A}{2}\left[\sin\frac{B}{2}\sin\frac{C}{2} - \cos\frac{B}{2}\cos\frac{C}{2}\right]$$

$$= 2R - 4R\sin\frac{A}{2}\cos\left(\frac{B+C}{2}\right)$$

$$= 2R - 4R\sin\frac{A}{2}\cos\left(\frac{B+C}{2}\right)$$

$$= 2R\left(1 - 2\sin^2\frac{A}{2}\right) = 2R\cos A = \text{L.H.S.}$$

Sol.3 $\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2} = \frac{s^2}{\Delta}$

$$\text{L.H.S.} = \frac{s(s-a)}{\Delta} + \frac{s(s-b)}{\Delta} + \frac{s(s-c)}{\Delta}$$

$$= \frac{s}{\Delta}(s - a + s - b + s - c)$$

$$= \frac{s}{\Delta}(3s - 2s) = \frac{s^2}{\Delta}$$

Sol.4 $\cos^2\frac{A}{2} + \cos^2\frac{B}{2} + \cos^2\frac{C}{2} = 2 + \frac{r}{2R}$

$$\text{L.H.S.} = \frac{1+\cos A}{2} + \frac{1+\cos B}{2} + \frac{1+\cos C}{2}$$

$$= \frac{3}{2} + \frac{1}{2}(\cos A + \cos B + \cos C)$$

$$= \frac{3}{2} + \frac{1}{2}\left(1 + 4\pi\sin\frac{A}{2}\right)$$

$$= 2 + \frac{r}{2R} = \text{R.H.S.}$$

Aliter :

$$\text{L.H.S.} = 1 + \left(\cos^2\frac{A}{2} - \sin^2\frac{B}{2}\right) + \cos^2\frac{C}{2}$$

$$= 1 + \sin\frac{C}{2}\cos\left(\frac{A-B}{2}\right) + 1 - \sin^2\frac{C}{2}$$

$$= 2 + \sin\frac{C}{2}\left[\cos\left(\frac{A-B}{2}\right) - \sin\frac{C}{2}\right]$$

$$= 2 + \sin\frac{C}{2}\left[\cos\left(\frac{A-B}{2}\right) - \cos\left(\frac{A+B}{2}\right)\right]$$

$$= 2 + \sin\frac{C}{2} \,.\, 2\sin\frac{A}{2}\sin\frac{B}{2}$$

$$= 2 + 2\pi\sin\frac{A}{2} = 2 + \frac{r}{2R} = \text{R.H.S.}$$

Sol.5

$$\text{L.H.S.} = \frac{\tan\frac{A}{2}}{(s-b)(s-c)} + \frac{\tan\frac{B}{2}}{(s-a)(s-c)} + \frac{\tan\frac{C}{2}}{(s-a)(s-b)}$$

$$= \frac{3r}{(s-a)(s-b)(s-c)} = \frac{3rs}{s(s-a)(s-b)(s-c)}$$

$$= \frac{3\Delta}{\Delta^2} = \frac{3}{\Delta}$$

Sol.6 $r_1 = r + r_2 + r_3$

$$\Rightarrow \frac{\Delta}{s-a} = \frac{\Delta}{s} + \frac{\Delta}{s-b} + \frac{\Delta}{s-c}$$

$$\Rightarrow \frac{1}{s-a} - \frac{1}{s} = \frac{1}{s-b} + \frac{1}{s-c}$$

$$\Rightarrow \frac{a}{s(s-a)} = \frac{a}{(s-b)(s-c)}$$

$\Rightarrow s^2 - as = s^2 - (b + c)s + bc$

$\Rightarrow bc = s(b + c - a)$

$\Rightarrow 2bc = (a + b + c)(b + c - a)$

$\Rightarrow 2bc = (b + c)^2 - a^2$

$\Rightarrow$ $2bc = b^2 + c^2 + 2bc - a^2$

$\Rightarrow$ $a^2 = b^2 + c^2$

$\Rightarrow$ ΔABC is right angled triangle

Sol.7 $2(2R)^2 = a^2 + b^2 + c^2$

$\Rightarrow$ $8R^2 = a^2 + b^2 + c^2$

$\Rightarrow$ $2 = \sin^2 A + \sin^2 B + \sin^2 C$

$\Rightarrow$ $2 = \sin^2 A + 1 - \cos^2 B + \sin^2 C$

$\Rightarrow$ $1 = \sin^2 A - \cos(B + C)\cos(B - C)$

$\Rightarrow$ $\cos^2 A = \cos A \cos(B - C)$

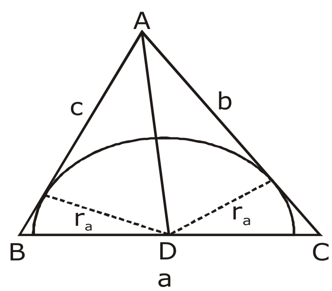
$\Rightarrow$ $\cos A[\cos(B - C) - \cos A] = 0$

$\Rightarrow$ $\cos A[\cos(B - C) + \cos(B + C)] = 0$

$\Rightarrow$ $2\cos A \cos B \cos C = 0$

$\Rightarrow$ $A = 90^\circ$ or $B = 90^\circ$ or $C = 90^\circ$

Sol.8 $\Delta = \Delta ABD + \Delta ACD$



$$\Rightarrow \Delta = \frac{1}{2} c\, r_a + \frac{1}{2} b\, r_a$$

$$\Rightarrow \Delta = \frac{r_a}{2}(b + c) \Rightarrow r_a = \frac{2\Delta}{b+c}$$

$$r_b = \frac{2\Delta}{c+a},\ r_c = \frac{2\Delta}{a+b}$$

$$\text{R.H.S.} = \frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c}$$

$$= \frac{(b+c)+(c+a)+(a+b)}{2\Delta} = \frac{2(a+b+c)}{2\Delta}$$

$$= \frac{2s}{\Delta} = \frac{2}{r}$$

Sol.9 $B = 3C$

$\because$ $A + B + C = 180^\circ$

$\Rightarrow A + 3C + C = 180^\circ$ { $C \in$ first quadrant}

$\Rightarrow A + 4C = 180^\circ$

by sine rule

$$\frac{a}{\sin A} = \frac{b}{\sin 3C} = \frac{c}{\sin C}$$

$b \sin C = c \sin 3C$

$\sin C\,[b - 3C - 4c \sin^2 C] = 0$

$b - 3c + 4c\sin^2 C = 0$ $\{\because \sin C \neq 0\}$

$$\Rightarrow \sin^2 C = \frac{3c-b}{4c}$$

$$\Rightarrow \cos^2 C = 1 - \frac{3c-b}{4c} = \frac{c+b}{4c}$$

$$\Rightarrow \cos C = \sqrt{\frac{b+c}{4c}} \qquad \{\because \cos C > 0\}$$

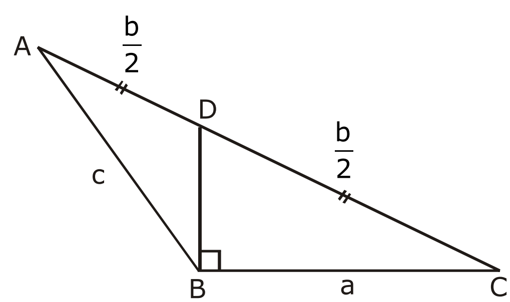
$\because$ $A + 4C = 180^\circ \Rightarrow \frac{A}{2} = 90^\circ - 2C$

$$\Rightarrow \sin\frac{A}{2} = \cos 2C = 2\cos^2 C - 1$$

$$= 2\left(\frac{b+c}{4c}\right) - 1 = \frac{b-c}{2c}$$

Sol.10 $BD = \dfrac{\sqrt{3}c}{4}$

In ΔBCD sine rule



$$\frac{\frac{b}{2}}{1} = \frac{\frac{\sqrt{3}c}{4}}{\sin C} \Rightarrow \frac{c}{\sin C} = \frac{2b}{\sqrt{3}}$$

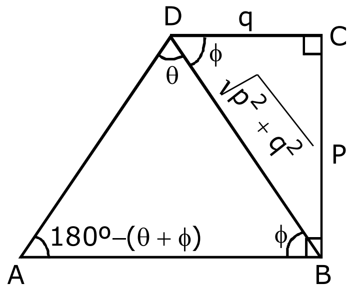
$\{\Delta ABC$ sine rule $\dfrac{c}{\sin C} = \dfrac{b}{\sin B}$

$$\frac{b}{\sin B} = \frac{2b}{\sqrt{3}} \Rightarrow \sin B = \frac{\sqrt{3}}{2}$$

B = 60º or 120º

$\because$ But B $\neq$ 60º $\therefore$ B = 120º {$\because$ B > 90º}

Sol.11 In ΔADB sine rule



$$\frac{AB}{\sin\theta} = \frac{\sqrt{p^2+q^2}}{\sin(180^o - \theta - \phi)}$$

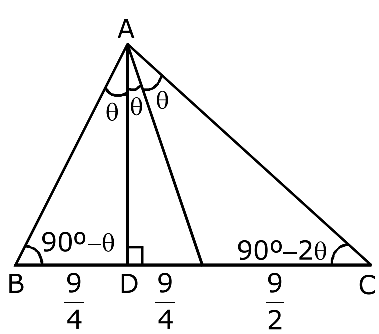
$$AB = \frac{\sin\theta\sqrt{p^2+q^2}}{\sin(\theta+\varphi)} = \frac{\sin\theta\sqrt{p^2+q^2}}{\sin\theta\cos\varphi + \cos\theta\sin\varphi}$$

$$= \frac{\sqrt{p^2+q^2}\sin\theta}{\left(\frac{q}{\sqrt{p^2+q^2}}\right)\sin\theta + \left(\frac{p}{\sqrt{p^2+q^2}}\right)\cos\theta}$$

$$= \frac{(p^2+q^2)\sin\theta}{p\cos\theta + q\sin\theta}$$

Sol.12 $\frac{BD}{CD} = \frac{1}{3}$

Use M–N theorem



(1 + 3) cot 90º = 1.cotθ – 3 cot 2θ

$\Rightarrow$ 0 = cot θ – 3 cot 2θ

$\Rightarrow$ cos θ = 3 cot 2θ

$\Rightarrow$ tan 2θ = 3 tan θ

$$\Rightarrow \frac{2\tan\theta}{1-\tan^2\theta} = 3\tan\theta$$

$\Rightarrow$ tan θ [2 – 3 + 3 tan$^2\theta$] = 0

$\Rightarrow$ tanθ [3tan$^2\theta$ –1] = 0

$\Rightarrow$ tan$\theta \neq 0$ or tan$^2\theta = \frac{1}{3}$

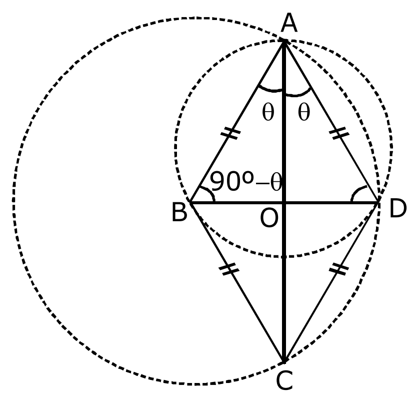
$\Rightarrow$ tan$\theta = \pm\frac{1}{\sqrt{3}}$ $\Rightarrow \theta$ = 30º, 150º

{$\because \theta = \frac{5\pi}{6}$ not possible} $\therefore$ $\theta = \frac{\pi}{6}$

Sol.13 AC = d_2, BD = d_1

R_2 = 25 R_1 = 12.5

In ΔABC, sine rule



$$\frac{d_1}{\sin 2\theta} = 2.(12.5)$$

d_1 = 25 sin2θ

& In ΔACD, by sine rule

$$\frac{d_2}{\sin(180-2\theta)} = 2.(25)$$

$\Rightarrow$ d_2 = 50 sin2θ

$d_2 = 2d_1$

Sine Rule, in ΔAOB

$$\frac{\left(\frac{d_1}{2}\right)}{\sin\theta} = \frac{\left(\frac{d_2}{2}\right)}{\sin(90^o-\theta)} \Rightarrow \frac{d_1}{\sin\theta} = \frac{d_2}{\cos\theta}$$

$$\Rightarrow \tan\theta = \frac{d_1}{d_2} \Rightarrow \tan\theta = \frac{1}{2}$$

$$\Rightarrow \sin 2\theta = \frac{2\left(\frac{1}{2}\right)}{1+\left(\frac{1}{2}\right)^2}$$

$$\Rightarrow \sin 2\theta = \frac{4}{5}$$

$$d_1 = 25 \times \frac{4}{5}$$

$$\Rightarrow d_1 = 20 \ \& \ d_2 = 40$$

Area of ABCD $= \frac{1}{2}\, d_1 \,.\, d_2 = \frac{1}{2}\, 20 \,.\, 40$

= 400 sq. unit

Sol.14 $a^2 + b^2 = 101\, c^2$

or $\sin^2 A + \sin^2 B = 101 \sin^2 C$

Now

$$\frac{\cot C}{\cot A + \cot B} = \frac{\frac{\cos C}{\sin C}}{\frac{\cos A}{\sin A} + \frac{\cos B}{\sin B}}$$

$$\Rightarrow \frac{\cos C \sin A \sin B}{\sin C \sin C (A+B)}$$

$$= \frac{\cos C(2\sin A \sin B)}{2\sin^2 C}$$

$$= \frac{\cos C[\cos(A-B) - \cos(A+B)]}{2\sin^2 C}$$

$$= \frac{\cos C[\cos(A-B) + \cos C]}{2\sin^2 C}$$

$$= \frac{-\cos(A+B)\cos(A-B) + \cos^2 C}{2\sin^2 C}$$

$$= \frac{-(\cos^2 A - \sin^2 B) + 1 - \sin^2 C}{2\sin^2 C}$$

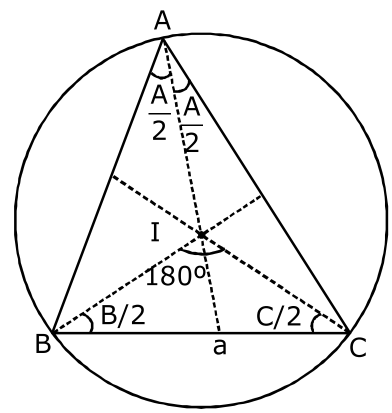
$$= \frac{\sin^2 B + (1-\cos^2 A) - \sin^2 C)}{2\sin^2 C}$$

$$= \frac{\sin^2 A + \sin^2 B - \sin^2 C}{2\sin^2 C}$$

$$= \frac{101\sin^2 C - \sin^2 C}{2\sin^2 C}$$

$$= \frac{101-1}{2} = \frac{100}{2} = 50$$

Sol.15 $\angle BIC = 180^o - \left(\frac{B+C}{2}\right) = 90^o + \frac{A}{2}$



$$BI = r \text{ cosec } \frac{B}{2}$$

$$CI = r \text{ cosec } \frac{C}{2}$$

In ΔBIC, Sine Rule

$$\frac{a}{\sin\left(90^o + \frac{A}{2}\right)} = \frac{r \operatorname{cosec}\frac{C}{2}}{\sin\frac{B}{2}} = \frac{r \operatorname{cosec}\frac{B}{2}}{\sin\frac{C}{2}} = 2x$$

$$\Rightarrow \frac{a}{\cos\frac{A}{2}} = \frac{r}{\sin\frac{B}{2}\sin\frac{C}{2}} = \frac{r}{\sin\frac{B}{2}\sin\frac{C}{2}} = 2x$$

$$\Rightarrow \frac{2a}{2\sin\frac{A}{2}\cos\frac{A}{2}} = \frac{r}{\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}$$

$$\Rightarrow \frac{2a}{\sin A} = 4R \Rightarrow a = 2R\sin A$$

$$\therefore 2x = \frac{a}{\cos\frac{A}{2}} \Rightarrow 2x = \frac{2R\sin A}{\cos\frac{A}{2}}$$

$\Rightarrow$ $x = 2R \sin \frac{A}{2}$

III'y $y = 2R \sin \frac{B}{2}$ & $z = 2R \sin \frac{C}{2}$

L.H.S. $= 4R^3 - R(x^2 + y^2 + z^2) - xyz$

$= 4R^3 - 4R^3 (\sin^2 \frac{A}{2} + \sin^2 \frac{B}{2} + \sin^2 \frac{C}{2})$

$- 8R^3 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$

$= 4R^3 [\cos^2 \frac{A}{2} - \sin^2 \frac{B}{2} - \sin^2 \frac{C}{2}]$

$- 8R^3 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$

$$= 4R^3 \left[\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) - \sin^2\frac{C}{2} - 2\pi \sin\frac{A}{2}\right]$$

$$=4R^3\left[\sin\frac{C}{2}\left\{\cos\left(\frac{A-B}{2}\right) - \cos\left(\frac{A+B}{2}\right)\right\} - 2\pi\sin\frac{A}{2}\right]$$

$$= 4R^3 \left[2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} - 2\pi\sin\frac{A}{2}\right] = 0 = \text{RHS}$$

Sol.16 $\frac{\cos A + 2\cos C}{\cos A + 2\cos B} = \frac{\sin B}{\sin C}$

$\Rightarrow$ $\sin C \cos A + 2 \sin C \cos C$

$= \sin B \cos A + 2 \sin B \cos B$

$\Rightarrow \cos A (\sin B - \sin C) + \sin 2B - \sin 2C = 0$

$\Rightarrow$ $\cos A \; 2\cos \left(\frac{B+C}{2}\right) \sin \left(\frac{B-C}{2}\right)$

$+ 2 \cos (B + C) \sin (B - C) = 0$

$$\Rightarrow 2\cos A \sin\left(\frac{B-C}{2}\right)\left[\cos\left(\frac{B+C}{2}\right) - 2\cos\left(\frac{B-C}{2}\right)\right] = 0$$

$\Rightarrow \cos A = 0$ or $\sin\left(\frac{B-C}{2}\right) = 0$

$\Rightarrow A = \frac{\pi}{2}$ or $B - C = 0 \Rightarrow B = C$

Sol.17 (i) $\frac{a}{\cos A} = \frac{b}{\cos B}$

$\Rightarrow$ $\frac{\sin A}{\cos A} = \frac{\sin B}{\cos B}$

$\Rightarrow$ $\tan A = \tan B \Rightarrow A = B$

now $2 \sin A \cos B = 2 \sin B \cos A$

$= \sin (A + B) - \sin (A - B)$

$\Rightarrow$ $2 \sin A \cos B = \sin C$ $\quad A = B$

(ii) $2 \sin A \cos B = \sin C$

$\sin (A + B) + \sin (A - B) = \sin C$

$\sin (A - B) = 0 \Rightarrow A = B$

$\Sigma \tan \frac{A}{2} \tan \frac{B}{2} = 1$

$\Rightarrow$ $\tan^2\frac{A}{2} + \tan\frac{A}{2}\tan\frac{C}{2} + \tan\frac{C}{2}\tan\frac{A}{2} = 1$

$\Rightarrow$ $\tan^2 \frac{A}{2} + 2 \tan \frac{A}{2} \tan \frac{C}{2} - 1 = 0$

(iii) $\tan^2\frac{A}{2} + 2 \tan \frac{A}{2} \tan \frac{C}{2} - 1 = 0$

$\Rightarrow$ $\left(\tan\frac{A}{2} + \tan\frac{C}{2}\right)^2 - \tan^2 \frac{C}{2} - 1 = 0$

$$\Rightarrow \frac{\sin^2\left(\frac{A+C}{2}\right)}{\cos^2\frac{A}{2}\cos^2\frac{C}{2}} = \sec^2 \frac{C}{2}$$

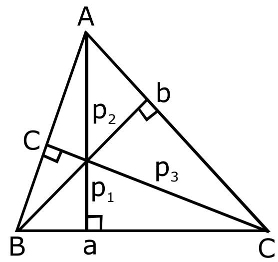
$$\Rightarrow \frac{\cos^2\frac{B}{2}}{\cos^2\frac{A}{2}} = 1 \Rightarrow \cos^2 \frac{B}{2} = \cos^2 \frac{A}{2}$$

$\Rightarrow$ $B = A$

$\Rightarrow$ $\tan B = \tan A$

$\Rightarrow$ $\frac{\sin A}{\cos A} = \frac{\sin B}{\cos B} \Rightarrow \frac{a}{\cos A} = \frac{b}{\cos B}$

Sol.18 $\Delta = \frac{1}{2}ap_1 \quad \frac{1}{p_1} = \frac{a}{2\Delta}$



III'y $\frac{1}{p_2} = \frac{b}{2\Delta}, \frac{1}{p_3} = \frac{c}{2\Delta}$

L.H.S $= \frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p_3} = \frac{a}{2\Delta} + \frac{b}{2\Delta} - \frac{c}{2\Delta}$

$= \frac{a+b-c}{2\Delta} = \frac{(a+b+c)(a+b-c)}{2(a+b+c)\Delta}$

$= \frac{(a+b)^2 - c^2}{2(a+b+c)\Delta} = \frac{(a^2+b^2-c^2)+2ab}{2(a+b+c)\Delta}$

$= \frac{2ab\cos C + 2ab}{2(a+b+c)\Delta} = \frac{ab(1+\cos C)}{(a+b+c)\Delta}$

$= \frac{2ab\cos^2\frac{C}{2}}{(a+b+c)\Delta}$ = R.H.S.

Sol.19 $\log a^2 = \log b^2 + \log c^2 - \log(2bc\cos A)$

$\Rightarrow \log a^2 = \log b^2c^2 - \log(b^2 + c^2 - a^2)$

$\Rightarrow \log a^2(b^2 + c^2 - a^2) = \log b^2c^2$

$\Rightarrow a^2b^2 + a^2c^2 - a^4 = b^2c^2$

$\Rightarrow b^2(a^2 - c^2) - a^2(a^2 - c^2) = 0$

$\Rightarrow (a^2 - c^2)(b^2 - a^2) = 0$

$\Rightarrow a = c$ or $b = a$

$\Rightarrow \Delta ABC$ is isosceles

Aliter :

$2a^2\, bc\cos A = b^2c^2 \Rightarrow \cos A = \frac{bc}{2a^2}$

$\Rightarrow \cos A = \frac{\sin B\sin C}{2\sin^2 A}$

$\Rightarrow 2\sin 2A\sin A = 2\sin B\sin C$

$\Rightarrow \cos A - \cos 3A = \cos(B - C) - \cos(B + C)$

$\Rightarrow \cos 3A + \cos(B - C) = 0$

$\therefore 3A + B - C = 180^\circ$

$2A + (A + B) - C = 180^\circ$

$2A + 180^\circ - C - C = 180^\circ$

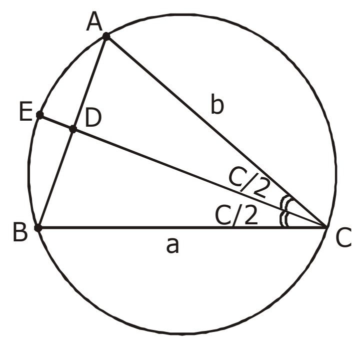
$2A = 2C \Rightarrow A = C$

$\Rightarrow \Delta ABC$ is isosceles

Sol.20 Power of point

$CD \times DE = AD \times DB$

$DE = \frac{AD \times DB}{CD}$



Now

$\frac{CD}{DE} = \frac{CD}{\left(\frac{AD\times DB}{CD}\right)} = \frac{(CD)^2}{AD\times DB} = \left\{\frac{AD}{DB} = \frac{b}{a}\right.$

$= \frac{\left(\frac{2ab}{a+b}\right)^2\cos^2\frac{c}{2}}{\left(\frac{bc}{a+b}\right)\times\left(\frac{ac}{a+b}\right)} = \frac{4a^2b^2\cos^2\frac{c}{2}}{abc^2}$

$= \frac{4ab}{c^2}\cos^2\frac{c}{2}$

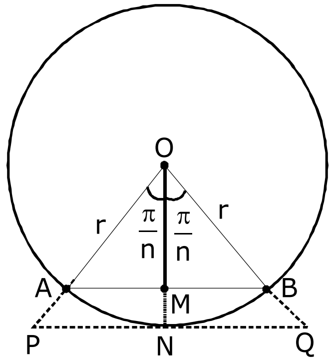
$\frac{CE}{DE} = \frac{CD + DE}{DE} = \frac{CD}{DE} + 1$

$= \frac{4ab}{c^2}\cos^2\frac{C}{2} + 1$

$= \frac{4ab}{c^2}\frac{s(s-c)}{ab} + 1 = \frac{2s(2s-2c)}{c^2} + 1$

$= \frac{(a+b)^2 - c^2 + c^2}{c^2} = \frac{(a+b)^2}{c^2}$

Sol.21 OM = $r \cos \frac{\pi}{n}$



$AB = 2r \sin \frac{\pi}{n}$

$A_1 = n \cdot \frac{1}{2} \cdot 2r \sin \frac{\pi}{n} \left(r \cos \frac{\pi}{n}\right)$

$A_1 = nr^2 \sin \frac{\pi}{n} \cos \frac{\pi}{n}$

Now $\frac{PN}{ON} = \tan \frac{\pi}{n}$

$\Rightarrow PN = r \tan \frac{\pi}{n}$

$\therefore B_1 = n \cdot \frac{1}{2} \cdot 2 \left(r \tan \frac{\pi}{n}\right) \cdot r$

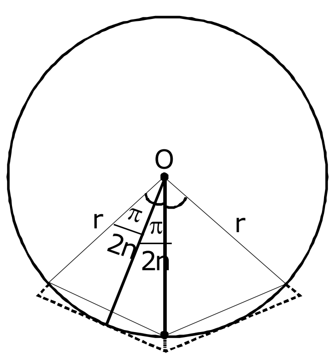
$B_1 = nr^2 \tan \frac{\pi}{n}$

III'y

$A_2 = 2nr^2 \sin \frac{\pi}{2n} \cos \frac{\pi}{2n}$

$A_2 = nr^2 \sin \frac{\pi}{n}$

$B_2 = 2nr^2 \tan \frac{\pi}{2n}$



(i) $A_2^2 = A_1 B_1$ is true

(ii) $\frac{2}{B_2} = \frac{1}{A_1} + \frac{1}{B_1}$ is also True

Sol.22 $\frac{R}{r_1} = \frac{\sqrt{2}}{\sqrt{2}+\sqrt{3}}$ & A = 90º = B + C

$\frac{r_1}{R} = 4 \sin \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2} = \frac{\sqrt{2}+\sqrt{3}}{\sqrt{2}}$

$\Rightarrow 4 \frac{1}{\sqrt{2}} \cos \frac{B}{2} \cos \frac{C}{2} = \frac{\sqrt{2}+\sqrt{3}}{\sqrt{2}}$

$\Rightarrow 2.2 \cos\frac{B}{2} \cos\left(\frac{\pi}{4} - \frac{B}{2}\right) = \sqrt{2} + \sqrt{3} \; \{B + C = \frac{\pi}{2}\}$

$= 2\left[\cos\frac{\pi}{4} + \cos\left(B - \frac{\pi}{4}\right)\right] = \sqrt{2} + \sqrt{3}$

$\Rightarrow \sqrt{2} + 2\cos\left(B - \frac{\pi}{4}\right) = \sqrt{2} + \sqrt{3}$

$\Rightarrow \cos\left(B - \frac{\pi}{4}\right) = \frac{\sqrt{3}}{2}$

$B - \frac{\pi}{4} = \frac{\pi}{6}$

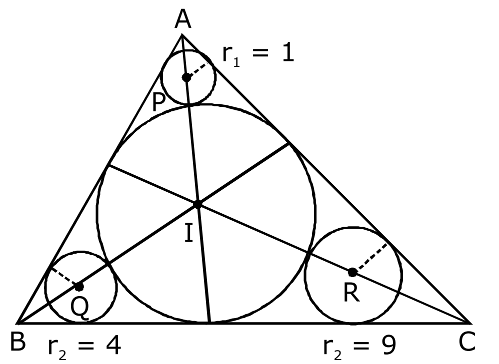
$B = \frac{5\pi}{12}, C = \frac{\pi}{12}$

$\frac{b}{c} = \tan B = 2 + \sqrt{3}$

Sol.23 $\frac{1}{AP} = \sin \frac{A}{2}$

$AP = \operatorname{cosec} \frac{A}{2}$

$AI = \operatorname{cosec} \frac{A}{2} + 1 + r$



$r \operatorname{cosec}\frac{A}{2} = \operatorname{cosec}\frac{A}{2} + 1 + r$

$\operatorname{cosec}\frac{A}{2} = \frac{r+1}{r-1}$

III'y $\operatorname{cosec}\frac{B}{2} = \frac{r+4}{r-4}$ & $\operatorname{cosec}\frac{C}{2} = \frac{r+9}{r-9}$

$\cot^2\frac{A}{2} = \operatorname{cosec}^2\frac{A}{2} - 1$

$= \frac{(r+1)^2-(r-1)^2}{(r-1)^2} = \frac{4r}{(r-1)^2}$

$\cot\frac{A}{2} = \frac{2\sqrt{r}}{(r-1)}$

III'y $\cot\frac{B}{2} = \frac{4\sqrt{r}}{(r-4)}$, $\cot\frac{C}{2} = \frac{6\sqrt{r}}{(r-9)}$

$\Sigma\cot\frac{A}{2} = \pi\cot\frac{A}{2}$

$\Rightarrow \frac{2\sqrt{r}}{(r-1)} + \frac{4\sqrt{r}}{(r-4)} + \frac{6\sqrt{r}}{(r-9)} = \frac{48r\sqrt{r}}{(r-1)(r-4)(r-9)}$

$\Rightarrow (r^2 - 13r + 36) + 2(r^2 - 10r + 9)$
$+ 3(r^2 - 5r + 4) = 24r$

$\Rightarrow 6r^2 - 48r + 66 = 24r$

$\Rightarrow r^2 - 8r + 11 = 4r$

$\Rightarrow r^2 - 12r + 11 = 0$

$\Rightarrow (r - 11)(r - 1) = 0$

$\Rightarrow r = 11$ or $r = 1$

$r = 11 \qquad \because r \neq 1$

Sol.24 $\Delta_{ABC} = 18$, $\quad a = c$

$\Delta_{BDF} = 2$

$DF = 2\sqrt{2}$

$\Delta = 18 = \frac{1}{2}ac\sin B \Rightarrow a^2\sin B = 36 \quad ...(i)$

$\Delta_{BDF} = 2 = \frac{1}{2}a\cos B \times c\cos B\sin B$

$\Rightarrow a^2\cos^2 B\sin B = 4$

$\Rightarrow \cos^2 B = \frac{4}{36} = \frac{1}{9} \quad ...(ii)$

$\Rightarrow \cos B = \pm\frac{1}{3}$

$\therefore \sin^2 B = 1 - \frac{1}{9} = \frac{8}{9}$

$\sin B = \frac{2\sqrt{2}}{3} \qquad \begin{cases}\sin B > 0 \\ \because \ B \in \text{I \& II quad.}\end{cases}$

$\frac{\Delta_{BDF}}{\Delta_{ABC}} = \frac{2}{18} = \frac{1}{9}$ then ratio between sides

is $\frac{DF}{AC} = \frac{1}{3}$

$\therefore \quad \frac{2\sqrt{2}}{b} = \frac{1}{3} \Rightarrow \quad b = 6\sqrt{2}$

by Sine Rule, $\frac{b}{\sin B} = 2R$

$\Rightarrow R = \frac{b}{2\sin B} = \frac{6\sqrt{2}}{2\left(\frac{2\sqrt{2}}{3}\right)}$

$R = \frac{9}{2}$